THE FINAL VALUE METHOD OF APPROXIMATING THE SOLUTION TO NON-LINEAR DIFFERENTIAL EQUATIONS WHICH ARE CONSTANT IN THE STEADY STATE

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THESIS

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ABSTRACT

The analysis of non-linear systems has frequently been a major problem to the engineer. The solution of system equations often requires either a computer or relatively complex numerical techniques. A straight-forward, relatively simple method is proposed herein which permits the engineer to satisfactorily approximate the exact solution to certain non-linear differential equations.

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I. INTRODUCTION

Systems described by non-linear differential equations often must be analyzed on a computer, using some numerical algorithm if the computer is of the digital type. The system variables are evaluated at specific points, and perhaps graphs drawn of these points. To discover the actual time solution to the system equations it is usually necessary to fit these points to some other numerical algorithm which will provide a curve fitting polynomial form of the exact solution. Such methods not only are not completely accurate but also are restrictive in that they require computer access; also, depending upon the complexity of the particular system, computer solution may require excessive time. These restrictions can sometimes be avoided if there is available a mathematical method to analyze the non-linear system.

Often, engineers are not as interested in exact solutions as they are in answering the questions:

1. What are the initial and steady-state values of the system variables?

2. What are the transient characteristics of the system? If the engineer can answer these questions he has, in effect, analyzed the system. The general nature of these questions suggests that the exact solution is not necessary if an approximate solution can be found which has essentially the same transient and steady-state responses.

II. DEVELOPMENT OF THE APPROXIMATION METHOD

Since the objective of an approximation method is to make the solution to non-linear equations readily obtainable, a starting point in finding such a method is to try to manipulate the equations into a linear, or at least less non-linear form. A method which has characteristics that can accomplish this purpose is the Laplace Transform equivalent to time equations.

A. DERIVATION OF THE SERIES FORM OF THE LAPLACE TRANSFORM

Baycura [ref. 1] proposed that the transform of certain non-linear terms could be found by using the series form of the Laplace Transform.

Recall that $X(s) = \int_0^{\infty} x(t)e^{-st}dt$ (II-1)

Expanding the right-hand side by integrating by parts,

$$\int_0^{\infty} xe^{-st}dt = x\int_0^{\infty} e^{-st}dt - x\int_0^{\infty}\int e^{-st}dtdt + x\int_0^{\infty}\int\int e^{-st}dtdtdt - \cdots$$

and evaluating x, its derivatives, and the integrals at the extreme points, it can be shown that the expansion becomes

$$\mathcal{L}\{x\} = \frac{x(0)}{s} + \frac{x(0)}{s^2} + \frac{x(0)}{s^3} + \frac{x(0)}{s^4} + \cdots \qquad \text{(II-2)}$$

Using this series form of the Laplace Transform, Brady [ref. 2] found the transform of $x^n(t)$.

$$\mathcal{L}\{x^n(t)\} = s^{n-1}x(s)^n \qquad \text{(II-3)}$$

Using similar methods a calculable final value term can be used as one of the extremes in the evaluation of the Laplace Integral to obtain a linearized series approximation to the Laplace Transform of non-linear terms.

B. OBTAINING THE FINAL VALUE

The method used to derive the series form of the Laplace Transform was to expand the Laplace Integral by integrating by parts and then evaluate each term at the extreme points, a necessary condition being that $xe^{-st} \to 0$ as $t \longrightarrow \infty$. The upper limit represents the steady-state, and for certain non-linear differential equations the steady-state, or final value, can be found.

For example, consider the non-linear differential equation

$$\dot{x} + x + x^2 = 1, \; x(0) = 0 \qquad \text{(II-4)}$$

In the steady-state, $\dot{x} = 0$, and

$$0 + x_f^2 + x_f = 1, \text{ where } x_f \text{ is the final value of } x.$$

Solving this quadratic equation produces the roots:

$$x_f = .615, \; -1.655$$

C. APPROXIMATING THE LAPLACE TRANSFORM OF THE NON-LINEAR TERM

After integrating by parts, the expanded form of the Laplace Transform of x^2 is

$$\mathcal{L}\{x^2\} = x\int_0^\infty xe^{-st}dt - \dot{x}\iint_0^\infty xe^{-st}dtdt + \ddot{x}\iiint_0^\infty xe^{-st}dtdtdt - \cdots \qquad \text{(II-5)}$$

Each term contains the term $\int_0^\infty xe^{-st}dt$ which is, by definition, the Laplace Transform, X(s). To obtain the series approximation each term in equation (II-5) is evaluated at its extreme points. In this manner the first term becomes

$$x\int_0^\infty xe^{-st}dt = (x_f - x_0)\cdot X(s)$$

Since integrating in the time domain corresponds to multiplying by $\frac{1}{s}$ in the s domain, the second term becomes

$$-\dot{x}\int\int_0^{\infty} xe^{-st}dtdt = -(\dot{x}_f - \dot{x}_0)\ X(s)dt = -\frac{(\dot{x}_f - \dot{x}_0)X(s)}{s}$$

It follows that similar evaluation of all the terms in the expansion will result in an infinite series of the form:

$$\mathcal{L}\{x^2\} = (x_f - x_0)X(s) - \frac{(\dot{x}_f - \dot{x}_0)X(s)}{s} + \frac{(\ddot{x}_f - \ddot{x}_0)X(s)}{s^2} - \cdots$$

$$= X(s)\left[(x_f - x_0) - \frac{(\dot{x}_f - \dot{x}_0)}{s} + \frac{(\ddot{x}_f - \ddot{x}_0)}{s^2} \mp \cdots\right] \qquad \text{(II-6)}$$

D. THE METHOD OF APPROXIMATION USING SUCCESSIVE TERMS IN THE INFINITE SERIES

The terms in the infinite series form given in equation (II-6) can be evaluated by taking derivatives of the original differential equation. Including more and more terms in the series will result in successive approximations to the transform of the non-linear term. Evaluation of these series terms is illustrated using equation (II-1). With the given conditions, $x_0 = 0$ and $\dot{x}_f = 0$,

$$\dot{x} + x + x^2 = 1 \qquad x_f = .615,\ x_0 = 1$$

$$\ddot{x} + \dot{x} + 2x\dot{x} = 0 \qquad \ddot{x}_0 = -1,\ \ddot{x}_f = 0$$

$$\dddot{x} + \ddot{x} + 2x\ddot{x} + 2\dot{x}^2 = 0 \qquad \dddot{x}_0 = -1,\ \dddot{x}_f = 0,\ \text{etc;.}$$

E. USE OF THE FINAL VALUE IN THE APPROXIMATE TRANSFORM

Using the series form of the transform of x^2, a final value term can be inserted to yield a linear equation in X(s) that is an approximation to the transform of x^2.

For example, consider a first approximation using only the first term of the series.

$$\mathcal{L}\{x^2\} = X(s)(x_f - x_0) = x_f X(s)$$

Transforming equation (II-4) term by term,

$$\mathcal{L}\{x^2\} = sX(s) - x_0 + X(s) + .615X(s) = \frac{1}{s}$$

$$X(s)(s + 1.615) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(s + 1.615)}$$

This equation can be expanded in partial fractions and the inverse of the s-domain solution found.

$$X(s) = \frac{1}{s(s + 1.615)} = \frac{A}{s} + \frac{B}{s + 1.615}$$

$$A = \frac{1}{s + 1.1615}\Big|_{s=0} = .618, \quad B = \frac{1}{s}\Big|_{s = -1.615} = -.618$$

$$X(s) = \frac{.618}{s} - \frac{.618}{s + 1.615} \qquad \text{(II-7)}$$

The time solution corresponding to equation (II-7) is

$$x(t) = .618 - .618\, e^{-1.615t}$$

This approximate solution has an initial value of zero and rises exponentially to a final value of .618. These values compare quite favorably with the given initial condition, $x_0 = 0$, and calculated final value, $x_f = .615$. A comparison between the approximate solution and the computer solution found on the SDS 9300 digital computer is shown in Table II-1 and Figure II-1. The comparison shows a close agreement between the two transient responses.

A second approximation to the transform would include the second term in the series form.

$$\mathcal{L}\{x^2\} = X(s)\left[(x_f - x_0) - \frac{(\dot{x}_f - \dot{x}_0)}{s}\right]$$

Substituting the values calculated for the terms in the series the second approximation to the transform of x^2 becomes $\mathcal{L}\{x^2\} = X(s)(.615-\frac{1}{s})$.

The effect of including derivative terms in the approximation is discussed in later sections.

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09486	.09216	.00270
0.2	.17911	.17058	.00853
0.3	.25269	.23231	.02038
0.4	.31600	.29408	.02192
0.5	.36979	.34329	.02650
0.6	.41501	.38349	.03152
0.7	.45268	.41847	.03421
0.8	.48382	.44822	.03560
0.9	.50940	.47354	.03586
1.0	.53032	.49509	.03521
2.0	.60837	.59358	.01479
3.0	.61701	.61314	.00387
4.0	.61793	.61718	.00075
5.0	.61802	.61781	.00021
10.0	.61800	.61800	.00000

TABLE II-1

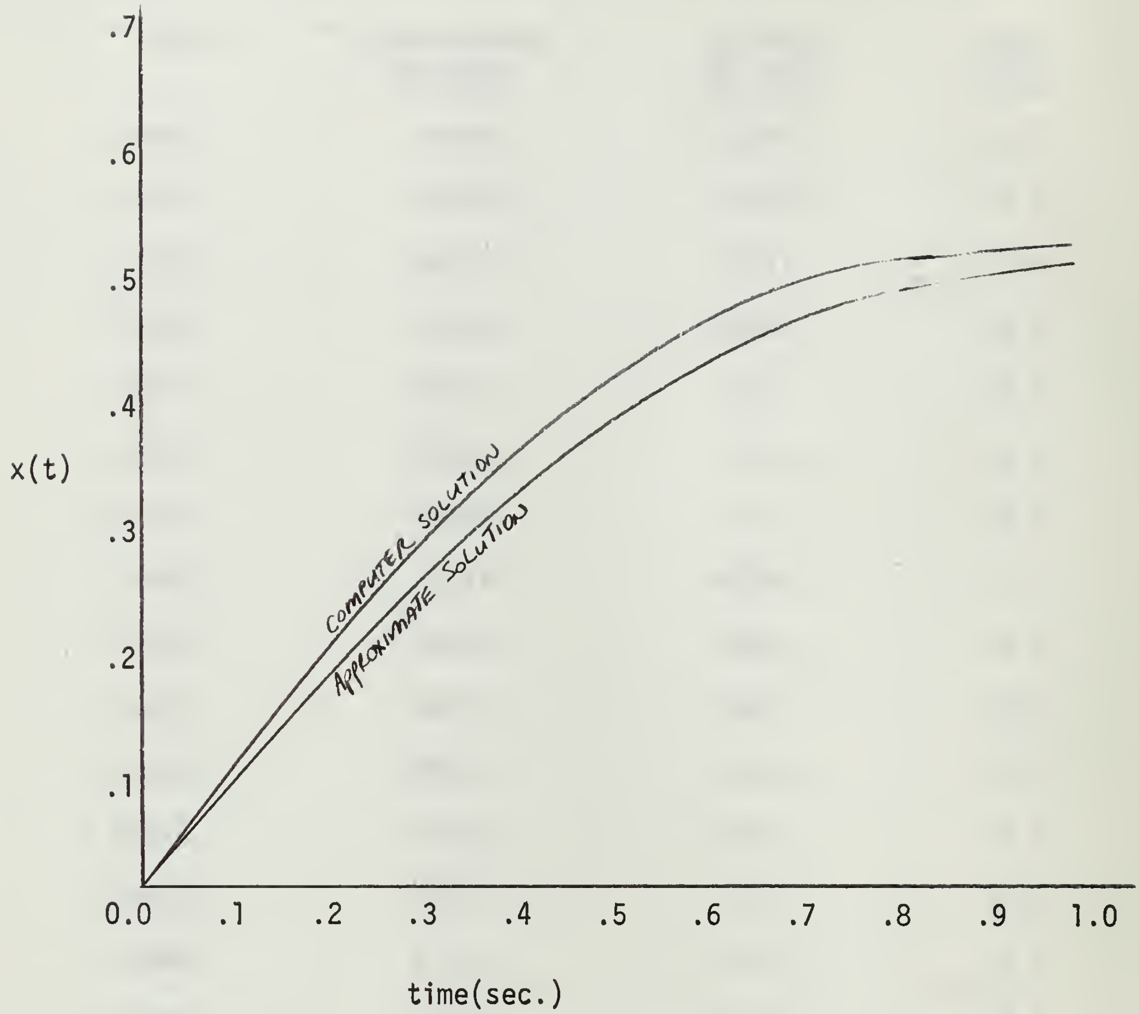
.7
.6
.5
.4
.3
.2
.1
x(t)
COMPUTER SOLUTION
APPROXIMATE SOLUTION
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
time(sec.)

FIGURE II-1

III. APPLICATION OF THE FINAL VALUE METHOD OF APPROXIMATION

The use of the final value method of approximating the solution to non-linear system equations can now be demonstrated on two equations containing first order non-linear terms.

A. APPLICATION TO A NON-HOMOGENEOUS NON-LINEAR EQUATION

Consider the case of vertical fall with air resistance. The governing equation of the system is

$$M\dot{v} = Mg - kv^2, \; v(0) = 0 \qquad \text{(III-1)}$$

where M is the mass of the falling object, g is the acceleration due to gravity, and k is a constant relating to the retarding force, which varies as the square of the velocity. Evaluating the system equation in the steady-state where $\dot{v} = 0$, the final value of the velocity is determined.

$$0 + kv_f^2 = Mg$$

$$v_f^2 = Mg/k$$

$$v_f = \sqrt{Mg/k}$$

The series approximation to the Laplace Transform of v^2 as given by equation (II-6) is

$$\mathcal{L}\{v^2\} = V(s)\left[(v_f - v_0) - \frac{(\dot{v}_f - \dot{v}_0)}{s} + \frac{(\ddot{v}_f - \ddot{v}_0)}{s^2} \cdots\right]$$

Having obtained the final value and knowing the series form of the approximate transform of v^2, successive approximations can be made to the solution of the system equation, adding successive terms with each approximation.

1. First Approximation.

Using the first term in the series form only,

$$\mathcal{L}\{v^2\} = v_f V(s) = \sqrt{Mg/k}\, V(s)$$

Transforming the system equation term by term,

$$\mathcal{L}\{M\dot{v} = Mg - kv^2\} = sV(s) + (k/M)v_f V(s) - g/s = 0$$

$$V(s) = \frac{g}{s(s + kv_f/M)}$$

Expanding in partial fractions,

$$V(s) = \frac{g}{s(s + kv_f/M)} = \frac{A}{s} + \frac{B}{(s + kv_f/M)}$$

$$V(s) = \frac{Mg/kv_f}{s} - \frac{Mg/kv_f}{(s + kv_f/M)}$$

Taking the inverse transform, and substituting the equation $v_f = \sqrt{Mg/k}$, the time solution is found.

$$v(t) = v_f(1 - e^{-(kv_f/M)t}) \qquad \text{(III-2)}$$

The exact solution can be obtained by separation of variables:

$$dv/dt = g - kv^2/M$$

$$\frac{dv}{g - kv^2/M} = dt$$

$$kM \quad \frac{dv}{Mg/k - v^2} = t$$

$$t = kM(1/2\ Mg/k)\ln \frac{(Mg/k + v)}{(Mg/k - v)}$$

$$e^2\ gk/M = \frac{v_f + v}{v_f - v}$$

$$v = v_f \frac{(1 - e^{-2\sqrt{gk/M}\,t})}{(1 + e^{-2\sqrt{gk/M}\,t})} \qquad \text{(III-3)}$$

As in the exact solution the velocity given in the approximate solution starts at zero and increases exponentially to v_f in the steady-state. The basic difference between the two solutions is the speed with which the velocity reaches the steady-state value. A comparison between the two solutions is shown in Table III-1 and Figure III-1, with $v_f = 36.4$ and $\sqrt{gk/M} = .27$.

2. <u>Second Approximation</u>.

In this Second Approximation the first two terms in the series form are used and the approximate Laplace Transform becomes:

$$\mathscr{L}\{v^2\} = V(s)\left[v_f - \frac{(\dot{v}_f - \dot{v}_0)}{s}\right] \qquad \text{(III-4)}$$

The added terms are evaluated remembering that the necessary condition for the final value method of approximation is that $\dot{v}_f = 0$. $\dot{V}_0$ can be calculated from the system equation.

$$\dot{v}_0 = gM/M - 0k$$

$$\dot{v}_0 = g$$

Substituting this value into equation (III-4),

$$\mathscr{L}\{v^2\} = V(s)\ [v_f + g/s]$$

After transforming the system equation term by term, a partial fraction expansion is used to find the s domain characteristic equation.

$$sV(s) + V(s)(k/M)(v_f + g/s) = g/s$$

$$V(s)\ \frac{(s + kv_f}{M} + \frac{kg/M)}{s} = g/s$$

$$V(s) = \frac{g}{(s + \sqrt{gk/M}\ s + gk/M)}$$

Expanding in partial fractions and taking the inverse transform, the time solution is found. With $L = k/M$,

$$V(s) = \frac{g}{(s + \sqrt{Lg/4} - j\sqrt{3Lg/4})(s + \sqrt{Lg/4} + j\sqrt{3Lg/4})}$$

$$= \frac{A}{(s + \sqrt{Lg/4} - j\sqrt{3Lg/4})} + \frac{A^*}{(s + \sqrt{Lg/4} + j\sqrt{3Lg/4})}$$

$$A = \frac{g}{j\sqrt{3Lg}} = \frac{g}{\sqrt{3Lg}} e^{-j90}, \quad A^* = \frac{g}{\sqrt{3Lg}} e^{j90}$$

$$v(t) = \sqrt{4/3}\sqrt{g/L}\, e^{-\sqrt{Lg/4}\, t} \sin \sqrt{3Lg/4}\; t \qquad \text{(III-5)}$$

It is apparent that v(t) has a steady-state value of zero. Also, this approximation has the transient characteristic of a damped sinusoidal, while the exact solution was essentially exponential with no oscillations.

3. Discussion of Results.

The first approximation led to a characteristic equation that was linear in s and was very easy to inversely transform to find the time solution. The second approximation contained a characteristic equation that was quadratic in s and a little more difficult to transform, but still relatively simple. The quadratic term was a direct result of the 1/s factor in the second term of the series form of the Laplace Transform of v^2. It follows that as succeeding terms are added to the approximation the characteristic equation will become of increasingly higher degree in s and more difficult to transform into the time domain. The requirement to solve difficult equations is contrary to the basic objective of finding a simple mathematical approximation. While the first approximation was exact in the steady-state and had the same general transient characteristics, the second approximation was very inaccurate, both in the steady-state and transient forms. If even more terms are added in an attempt to find the best possible approximation, the resulting s domain equations become very difficult to solve and transform.

TIME	EXACT SOLUTION	APPROXIMATE SOLUTION	ERROR
.00	.00000	.00000	.00000
.01	.09828	.09815	.00013
.02	.19656	.19603	.00053
.03	.29483	.29365	.00118
.04	.39310	.39100	.00210
.05	.49137	.48810	.00327
.06	.58963	.58493	.00370
.07	.68788	.68150	.00638
.08	.78612	.77781	.00831
.09	.88435	.87386	.01049
.10	.98256	.96965	.01291
.20	1.96369	1.91347	.05022
.30	2.83215	2.74804	.08411
.40	3.91599	3.72636	.22963
.50	4.88436	4.59674	.28762
1.00	9.59594	9.35320	.24274
2.00	15.75303	13.56130	2.19173
3.00	24.37309	20.63852	3.73457
4.00	28.87245	24.36801	4.50444
5.00	31.81554	27.21503	4.60051
10.00	36.05460	33.88677	2.16783
11.1	36.40000	34.62811	1.77189
13.7	36.40000	36.40000	0.00000

TABLE III-1

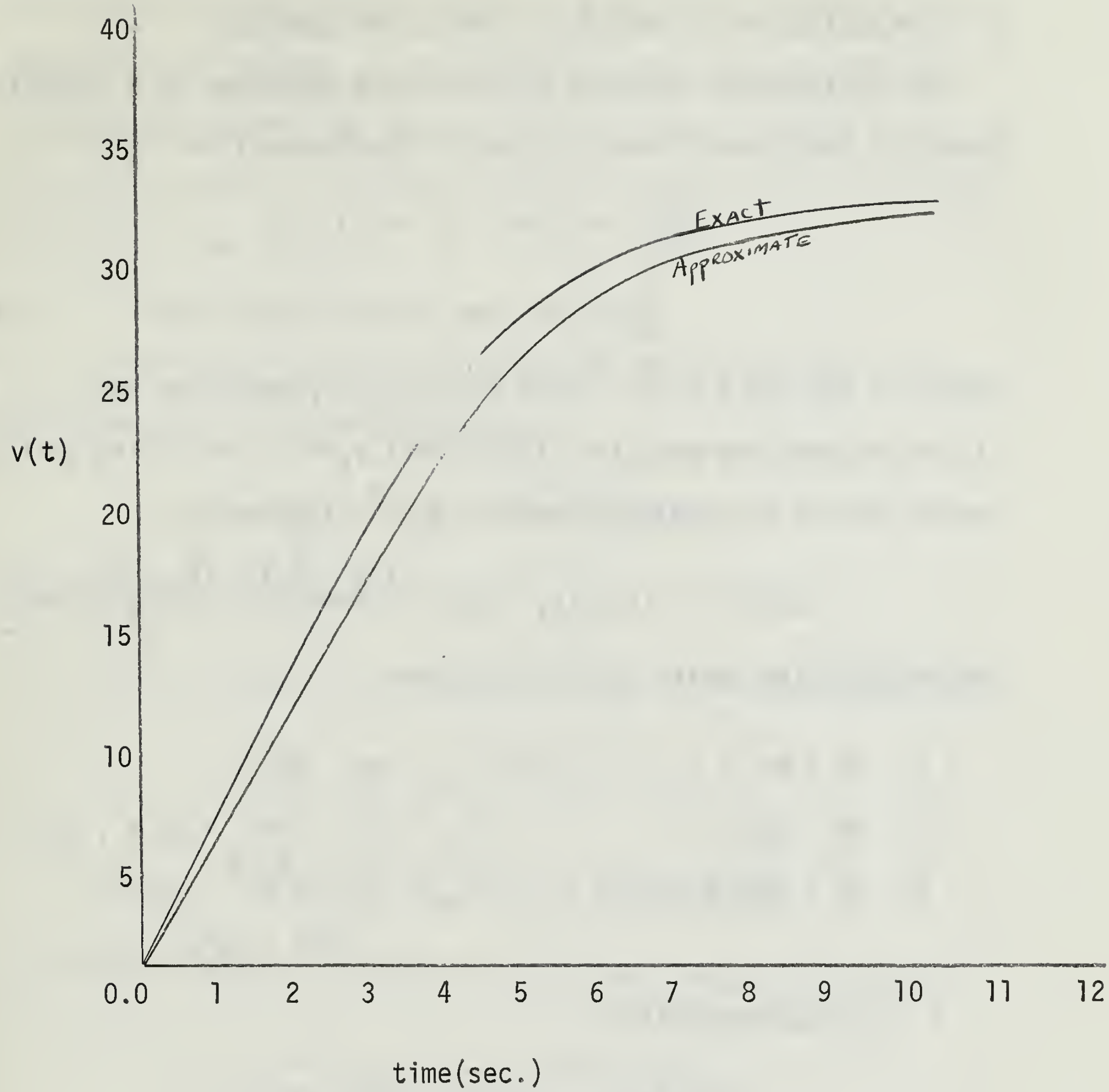
40
35
30
25
v(t)
20
15
10
5
0.0
1
2
3
4
5
6
7
8
9
10
11
12
EXACT
APPROXIMATE
time(sec.)

FIGURE III-1

B. APPLICATION TO A HOMOGENEOUS NON-LINEAR EQUATION

The differential equation governing the discharge of a capacitor through a non-linear diode is given by Cunningham [ref. 3].

$$C\frac{de}{dt} + ae + be = 0, \quad e(0) = V$$

$$\frac{de}{dt} = -Ae - Be, \qquad \text{(III-6)}$$

where A = a/C and B = b/C. With the necessary condition that $e_f = 0$, it can be seen from equation (III-6) that $e_f = 0$. Recalling that the series form of the Laplace Transform of e^2 is given by

$$\mathcal{L}\{e^2\} = E(s)\left[(e_f - e_0) - \frac{(\dot{e}_f - \dot{e}_0)}{s} + \frac{(\ddot{e}_f - \ddot{e}_0)}{s^2} \cdots\right],$$

the terms in the series can be calculated.

$$\dot{e} + Ae + Be^2 = 0 \qquad \dot{e}_0 = -Av - Bv^2$$

$$\ddot{e} + A\dot{e} + 2Be\dot{e} = 0 \qquad \ddot{e}_0 = A^2V + 3ABV^2 + 2B^2V^3, \ \ddot{e}_f = 0$$

$$\dddot{e} + A\ddot{e} + 2Be\ddot{e} + 2B\dot{e}^2 = 0 \qquad \dddot{e}_0 = -A^3V - 7A^2BV^2 - 10AB^2V^3 - 4B^3V^4 - 2B^3V^4, \ \dddot{e}_f = 0$$

1. First Approximation.

$$\mathcal{L}\{e^2\} = (e_f - e_0)E(s) = -VE(s)$$

Transforming equation (III-6) term by term,

$$\mathcal{L}\{e + Ae + Be^2 = 0\} = sE(s) - V + AE(s) - BVE(s) = 0 \qquad \text{(III-7)}$$

$$E(s) = \frac{V}{s + A - BV}$$

Taking the inverse transform the approximate time solution is obtained

$$e(t) = Ve^{-At}e^{BVt} \qquad \text{(III-8)}$$

Cunningham [ref. 3] gives the true solution as

$$e(t) = \frac{Ve^{-At}}{1 - BV/A(e^{-At}-1)} \qquad \text{(III-9)}$$

Both solutions have initial values of V and final values of zero. The transient response is an exponential decay to 0 in both cases. A comparison between the two solutions is shown in Table III-2. It can be seen from the solutions that the approximation will be best for BV much less than A.

2. Second Approximation.

Including the first two terms of the series form in the approximation, the transform becomes

$$\mathcal{L}\{e^2\} = \left[(e_f - e_0) - \frac{(\dot{e}_f - \dot{e}_0)}{s}\right] E(s)$$

After transforming equation (III-6),

$$\{\dot{e} + Ae + Be^2 = 0\} = sE(s) - V + AE(s) + B(-V-\frac{(AV+BV^2)}{s}) = 0$$

$$E(s)(s + A - BV - \frac{(ABV + B^2V^2)}{s}) = V$$

$$E(s) = \frac{Vs}{s^2 + (A - BV)s - (ABV + B^2V^2)}$$

Expanding in partial fractions, the s domain solution is found.

$$E(s) = \frac{M}{s + \frac{(A-BV)}{2} + \frac{\sqrt{A^2 - 2BV + 4ABV + 5B^2V^2}}{2}} + \frac{N}{s + \frac{(A-BV)}{2} - \frac{\sqrt{A^2 - 2BV + 4ABV + 5B^2V^2}}{2}}$$

It is apparent that the denominator of the second term in the right-hand side of the partial fraction expansion will, when transformed, cause the time solution to be unstable as the steady-state is approached. With the constants A = 1, V = 1, and B = .5, the time solution was found to be

$$e(t) = .6052 \exp(-1.15t) + .3946 \exp(.75t)$$

The second term is unstable.

3. Discussion of Results.

It can be shown that a third approximation to the solution using the first three terms in the series form of the transform of e^2, with A = 1, B = .5, and V = 1, with result in the following s domain characteristic equation:

$$E(s) = \frac{s^2}{s^3 + .5s^2 - .75s - 1.5}$$

The corresponding time solution is

$$e(t) = .22e^{.9965t} + Ke^{-.45t} \cos(1.2t + \theta)$$

As in the second approximation, this solution contains an unstable term. These results can be predicted by examining the characteristic equation in both cases. The presence of negative terms in these equations leads to unstable terms in the time solutions, since negative terms are contrary to known stability criteria. Furthermore, as more and more terms of the series approximation to the transform of the non-linear term are added, the n^{th} order characteristic polynomial equation will contain at least one negative term if the $(n-1)^{th}$ order equation contains at least one.

C. DISCUSSION OF THE METHOD OF SUCCESSIVE APPROXIMATION

In both examples of the use of successive approximations, including only the first term in the approximation resulted in a satisfactory comparison to the exact solution while adding more terms not only made the solution more difficult, but also introduced unstable terms in one case, and an inaccurate solution in the other. In the following sections the final value method will be applied to more examples, but only the first approximation will be used.

TIME	EXACT SOLUTION	APPROXIMATE SOLUTION	ERROR
.00	1.00000	1.00000	.00000
.01	.98515	.99501	.00986
.02	.97059	.99005	.01946
.03	.95631	.98511	.02880
.04	.94232	.98020	.03782
.05	.92859	.97531	.04672
.06	.91512	.97045	.06467
.07	.90191	.96561	.06370
.08	.88894	.96079	.07185
.09	.87622	.95600	.08918
.10	.86374	.95123	.08749
.20	.75069	.90484	.25415
.30	.65583	.86071	.20488
.40	.57546	.81873	.24327
.50	.50682	.77880	.27198
1.00	.27953	.60653	.32700
2.00	.09449	.36788	.27339
3.00	.03375	.22313	.18938
4.00	.01229	.13534	.12305
5.00	.00450	.08208	.07758
10.00	.00003	.00708	.00705

TABLE III-2

A = 1.0, B = 0.5, V = 1.0

IV. A DETAILED ANALYSIS OF THE APPLICATION OF THE FINAL VALUE METHOD USING FIRST APPROXIMATIONS ONLY.

In the following sections the results obtained from the application of the approximation method to the equation

$$\dot{x} + x + Ax^3 = 1, \ x(0) = 0 \qquad \text{(IV-1)}$$

will be shown. The effect of varying A, the coefficient of the non-linear term, will be studied by comparing the approximate solutions to solutions obtained on the SDS 9300 Digital Computer using Subroutine RKAM. In addition, the solutions obtained when a second derivative term was added to the same equation and A again varied will be analyzed in the same manner.

A. APPLICATION TO A SECOND ORDER NON-LINEAR EQUATION

To illustrate the application of the approximation method to equation (IV-1) consider this equation with A = 1:

$$\dot{x} + x + x^3 = 1, \ x(0) = 0 \qquad \text{(IV-2)}$$

Transforming the equation term by term,

$$\mathcal{L}\{\dot{x} + x + x^3 = 1\} = sX(s) - x(0) + X(s) + \quad \{x^3\} = 1/s$$

1. Finding The Approximate Laplace Transform of x^3.

The approximate transform of the second order non-linear term, x^3, was found as follows:

by definition, $\mathcal{L}\{X^3\} = x^3e^{-st}dt$

Expanding by integrating by parts,

$$\int_0^\infty x^3e^{-st}dt = x^2\int_0^\infty xe^{-st}dt - (x^2)\iint_0^\infty xe^{-st}dtdt + (x^2)\iiint_0^\infty xe^{-st}dtdtdt - \ldots$$

Since $\int xe^{-st}dt = X(s)$, and integrating in time corresponds to multiplying by 1/s in the s domain, evaluating x^2 and its derivatives at the extremes results in the series,

$$\mathcal{L}\{x^3\} = (x_f^2 - x_0^2)X(s) - (\dot{x}_f^2 - \dot{x}_0^2)\frac{X(s)}{s} + (\ddot{x}_f^2 - \ddot{x}_0^2)\frac{X(s)}{s^2} - \cdots$$

Rearranging terms,

$$\mathcal{L}\{x^3\} = X(s)\left[(x_f^2 - x_0^2) - \frac{(\dot{x}_f^2 - \dot{x}_0^2)}{s} + \frac{(\ddot{x}_f^2 - \ddot{x}_0^2)}{s^2} - \cdots\right] \qquad \text{(IV-5)}$$

For a first approximation all derivatives of x^2 are neglected and the linearized approximation to the Laplace Transform of x^3 becomes

$$\mathcal{L}\{x^3\} = (x_f^2 - x_0^2)X(s) \qquad \text{(IV-6)}$$

Recalling the required condition that $\dot{x}_f = 0$, the final value, x_f, can be found.

$$0 + x + x^3 = 1$$

$$x^3 + x - 1 = 0 \qquad \text{(IV-7)}$$

The cubic equation has three roots, one real and two complex. Selecting the real one,

$$x_f = .682$$

Using this value of x_f, equation (IV-6) becomes

$$\mathcal{L}\{x^3\} = .4651X(s).$$

2. <u>Alternate Methods of Finding The Approximate Transform.</u>

To further illustrate the obtaining of the approximate transform of the non-linear term, two different approaches will be used. Equation (IV-3) can also be expanded as

$$\int_0^\infty x^3e^{-st}dt = x\int_0^\infty x^2e^{-st}dt - x\int_0^\infty\!\!\int x^2e^{-st}dtdt + x\int_0^\infty\!\!\int\!\!\int x^2e^{-st}dtdtdt - \cdots$$

Using the Laplace Transform of x^2 found by Brady [ref. 2],

$$x^2 e^{-st} dt = sX^2(s),$$

and evaluating the terms in the expansion as before,

$$\mathcal{L}\{x^3\} = (x_f - x_0)sX^2(s) - \frac{(\dot{x}_f - \dot{x}_0)}{s} sX^2(s) + \frac{(\ddot{x}_f - \ddot{x}_0)}{s^2} sX^2(s) - \cdots$$

Neglecting derivatives and rearranging,

$$\mathcal{L}\{x^3\} = sX^2(s)x_f \qquad \text{(IV-8)}$$

Now, expanding X(s) in its infinite series form,

$$X(s) = \frac{(x_f - x_0)}{s} - \frac{(\dot{x}_f - \dot{x}_0)}{s^2} + \frac{(\ddot{x}_f - \ddot{x}_0)}{s^3} - \cdots$$

If derivatives are again neglected,

$$X(s) = \frac{x_f - x_0}{s} = \frac{x_f}{s}$$

and equation (IV-8) becomes

$$\mathcal{L}\{x^3\} = \frac{sx_f x_f X(s)}{s} = x_f^2 X(s)$$

The approximate transform thus obtained is identical to equation (IV-6).

The final approach will be to take the transform of x^3 directly, using results found by Brady [ref. 2]:

$$\mathcal{L}\{x^3\} = s^2X^3(s) = s^2X(s)X^2(s) \qquad \text{(IV-9)}$$

Writing $X^2(s)$ as the product of two infinite series,

$$X^2(s) = \left[\frac{(x_f-x_0)}{s} - \frac{(\dot{x}_f-\dot{x}_0)}{s^2} + \frac{(\ddot{x}_f-\ddot{x}_0)}{s^3} - \cdots\right]\left[\frac{(x_f-x_0)}{s} - \frac{(\dot{x}_f-\dot{x}_0)}{s^2} + \cdots\right]$$

Neglecting derivatives and products containing derivatives,

$$X^2(s) = \frac{x_f x_f}{s\ s} = \frac{x_f^2}{s^2}$$

Substituting this result into equation (IV-9), the approximation becomes,

$$\mathcal{L}\{x^3\} = \frac{s^2 X(s) x_f^2}{s^2} = x_f^2 X(s)$$

Again, the approximate transform obtained is identical to equation (IV-6).

3. <u>Using The Approximate Transform To Find The Solution.</u>

Completing the term by term transform of equation (IV-1)

$$\mathcal{L}\{\dot{x} + x + x^3 = 1\} = sx(s) - x(0) + X(s) + x_f^2 X(s) = 1/s$$

With x(0) = 0, this equation becomes

$$X(s)(s + 1 + .4651) = 1/s$$

$$X(s) = \frac{1}{s(s + 1.4651)}$$

Taking the inverse transform, the approximate solution is found.

$$X(s) = \frac{}{s(s + 1.4651)} = \frac{A}{s} = \frac{B}{(s + 1.4651)}$$

$$A = \frac{1}{1.4651}, \qquad B = \frac{-1}{1.4651}$$

$$X(s) = \frac{.6825}{s} - \frac{.6825}{s + 1.4651}$$

$$x(t) = .6825 - .6825e^{-1.4651t}$$

A comparison between the approximate solution and the computer solution is shown in Table IV-3.

4. <u>Variation of the Coefficient of the Non-linear Term.</u>

Extending the methods developed to the solution of the general equation,

$$\dot{x} + x + Ax^3 = 1, \; x(0) = 0,$$

the final values are found by solving the cubic equation,

$$x^3 + \frac{x}{A} - \frac{1}{A} = 0$$

With A = .5, this equation becomes

$$x^3 + 2x - 2 = 0.$$

The solution to the cubic equation of the form,

$$x^3 + ax + b = 0$$

can be found [ref. 4]:

Letting $A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$, $B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}$,

then the roots are

$$x = A + B,\ -\frac{(A+B)}{2} + \frac{(A-B)}{2}\sqrt{-3},\ \ -\frac{(A+B)}{2} - \frac{(A-B)}{2}\sqrt{-3}$$

Choosing the real root, x = A + B.

With A = .5, the auxiliary equations are:

$$A = \sqrt[3]{\frac{2}{2} + \sqrt{\frac{4}{4} + \frac{8}{27}}}, \quad B = \sqrt[3]{\frac{2}{2} - \sqrt{\frac{4}{4} + \frac{8}{27}}}$$

and $x_f = .7713$, $x_f^2 = .5949$.

Similarly, for other values of A, the values of x_f and x_f^2 were found to be as follows:

A	x_f	x_f^2
.5	.7713	.5949
.9	.6972	.4861
1.0	.6825	.4651
1.1	.6709	.4500
1.5	.6257	.3914
2.0	.5904	.3481
5.0	.4847	.2349

The general characteristic equation was found by transforming the general equation term by term, with the initial condition, x(0) = 0.

$$\mathcal{L}\{\dot{x} + x + Ax^3 = 1\} = sX(s) + X(s) + Ax_f^2X(s) = 1/s$$

$$X(s) = \frac{}{x(x + 1 + Ax_f^2)} = \frac{B}{s} + \frac{C}{(s + 1 + Ax_f^2)}$$

$$B = \frac{1}{1 + Ax_f^2}, \quad C = \frac{-1}{1 + Ax_f^2}$$

The generalized time solution (approximate) is then

$$x(t) = \frac{1}{1 + Ax_f^2} - \frac{1}{1 + Ax_f^2} e^{-(1 + Ax_f^2)t}$$

With the same values of A, the time solutions were found to be as follows:

A	x(t)
.5	.7708 - .7708 exp (-1.2975t)
.9	.6905 - .6956 exp (-1.4375t)
1.0	.6825 - .6825 exp (-1.4651t)
1.1	.6689 - .6689 exp (-1.4950t)
1.5	.6300 - .6300 exp (-1.5871t)
2.0	.5896 - .5896 exp (-1.6962t)
5.0	.4600 - .4600 exp (-2.1745t)

Tables IV-1 through IV-7 show comparisons between the approximate solutions and the solutions found on the digital computer for the several values of A. For all values of A the approximations were quite satisfactory. In each case the approximate solution rose slightly slower to the steady-state. For the worst case, A = .5, the approximate steady-state value differed from the computer solution bt 2.6%. The maximum difference in the transient values was approximately 10%, but typical differences were much less.

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09515	.09379	.00136
0.2	.18112	.17618	.00494
0.3	.25851	.24853	.00998
0.4	.32784	.31209	.01575
0.5	.38958	.36790	.02168
0.6	.44418	.41693	.02725
0.7	.49215	.45999	.03216
0.8	.53401	.49781	.03620
0.9	.57030	.53103	.03927
1.0	.60158	.56021	.04137
2.0	.74303	.71362	.02941
3.0	.76665	.75508	.01157
4.0	.77027	.76651	.00376
5.0	.77082	.76936	.00118
6.0	.77090	.77048	.00042
7.0	.77091	.77071	.00021
8.0	.77092	.77078	.00014
9.0	.77092	.77079	.00013
10.0	.77092	.77079	.00012

TABLE IV-1
A = 0.5

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09514	.09314	.00200
0.2	.18100	.17381	.00719
0.3	.25798	.24367	.01431
0.4	.32640	.30418	.02122
0.5	.38656	.35659	.02997
0.6	.43884	.40198	.03686
0.7	.48376	.44130	.04246
0.8	.52193	.47535	.04658
0.9	.55402	.50484	.04918
1.0	.58075	.53038	.05037
2.0	.68374	.65636	.02738
3.0	.69499	.68628	.00871
4.0	.69612	.69339	.00273
5.0	.69623	.69507	.00116
6.0	.69624	.69538	.00076
7.0	.69624	.69557	.00067
8.0	.69624	.69559	.00065
10.0	.69624	.69560	.00064

TABLE IV-2
A = 0.9

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09514	.09301	.00213
0.2	.18097	.17335	.00762
0.3	.25785	.24274	.01511
0.4	.32605	.30267	.02338
0.5	.38583	.35443	.03140
0.6	.43756	.39914	.03842
0.7	.48176	.43776	.04400
0.8	.47111	.47111	.04796
0.9	.55022	.49992	.05030
1.0	.57596	.52481	.05115
2.0	.67181	.64606	.02575
3.0	.68138	.67408	.00730
4.0	.68224	.68005	.00169
5.0	.68233	.68205	.00028
6.0	.68233	.68240	.00007
7.0	.68233	.68248	.00015
8.0	.68233	.68249	.00016

TABLE IV-3
A = 1.0

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09513	.09288	.02259
0.2	.18094	.17287	.00807
0.3	.25772	.24175	.01597
0.4	.32569	.30106	.02453
0.5	.38508	.35214	.03294
0.6	.43626	.39613	.04013
0.7	.47976	.43401	.04575
0.8	.51626	.46662	.04964
0.9	.54650	.49471	.05179
1.0	.57131	.51890	.05241
2.0	.66073	.63526	.02547
3.0	.66891	.66136	.00755
4.0	.66960	.66721	.00239
5.0	.66966	.66852	.00114
6.0	.66966	.66881	.00085
7.0	.66966	.66888	.00078
8.0	.66966	.66890	.00076

TABLE IV-4
A = 1.1

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09513	.09246	.00267
0.2	.18082	.17134	.00948
0.3	.25720	.23865	.01855
0.4	.32438	.29609	.02819
0.5	.38218	.34509	.03709
0.6	.43126	.38690	.04436
0.7	.47211	.42258	.04953
0.8	.50556	.45302	.05254
0.9	.53256	.47899	.05356
1.0	.55406	.50115	.05291
2.0	.62321	.60365	.01956
3.0	.62787	.62461	.00326
4.0	.62816	.62890	.00070
5.0	.62818	.62877	.00159
6.0	.62818	.62995	.00177
7.0	.62818	.62999	.00181
8.0	.62818	.63000	.00182

TABLE IV-5
A = 1.5

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09511	.09199	.03129
0.2	.18067	.16962	.01105
0.3	.25655	.23514	.02141
0.4	.32255	.29044	.02311
0.5	.37867	.33712	.04155
0.6	.42529	.37651	.04878
0.7	.46315	.40875	.05340
0.8	.49329	.43781	.05548
0.9	.51687	.46149	.05538
1.0	.53506	.48148	.04356
2.0	.58710	.56977	.01733
3.0	.58963	.58596	.00367
4.0	.58975	.58893	.00082
5.0	.58975	.58948	.00027
6.0	.58975	.58958	.00027

TABLE IV-6
A = 2.0

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00000	.00000
0.1	.09505	.08990	.00515
0.2	.17977	.16223	.01754
0.3	.25276	.22042	.03234
0.4	.31280	.26724	.04556
0.5	.35978	.30491	.05487
0.6	.39484	.33522	.05962
0.7	.41999	.35961	.06038
0.8	.43748	.37923	.05825
0.9	.44938	.39501	.05437
1.0	.45734	.40771	.04963
1.5	.47075	.44237	.02838
2.0	.47231	.45406	.01325
2.5	.47249	.45800	.01449
3.0	.47251	.45932	.01319
3.5	.47251	.45977	.01274
4.0	.47251	.45992	.01159
4.5	.47251	.45997	.01254
5.0	.47251	.45999	.01252
5.5	.47251	.46000	.01251
10.0	.47251	.46000	.01251

TABLE IV-7
A = 5.0

B. THE EFFECT OF ADDING A SECOND DERIVATIVE TERM TO THE NON-LINEAR EQUATION

A second derivative term was added to equation (IV-1) and the approximate transform was used to find an approximate solution to the resulting equation. The equation,

$$\ddot{x} + \dot{x} + x + Ax^3 = 1, \; x(0) = 0, \qquad \text{(IV-11)}$$

still contained a second-order non-linear term, x^3.

1. Use of the Final Value Method to Find the Approximate Solution.

As in Part A the solution method will be illustrated on the example equation with A = 1:

$$\ddot{x} + \dot{x} + x + x^3 = 1, \; x(0) = 0 \qquad \text{(IV-12)}$$

The final value is the same as the one found for equation (IV-1), since x_f will also be zero in the steady-state. The approximate transform of the second-order non-linear term is also the same since its derivation was independent of the system equation.

Transforming equation (IV-12) term by term, which x(0) = 0,

$$\mathcal{L}\{\ddot{x} + \dot{x} + x + x^3 = 1\} = s^2X(s) + sX(s) + X(s) + x_f^2X(s) = 1/s$$

The s domain characteristic equation is then

$$X(s) = \frac{1}{s(s^2 + s + 1.4651)}$$

Expanding in partial fractions the s domain solution can be found.

$$\frac{1}{s(s^2 + s + 1.4651)} = \frac{A}{s} + \frac{B}{s + .5 - j1.102} + \frac{B^*}{s + .5 + j1.102}$$

$$A = .6825$$

$$B = -.3600e^{-j24^\circ\ 23'}$$

$$X(s) = \frac{.6825}{s} + \frac{-.36e^{-j24^\circ 23'}}{s + .5 - j1.102} + \frac{-.36e^{j24^\circ 23'}}{s + .5 + j1.102}$$

The corresponding time solution was found to be

$$x(t) = .6825 - .7200e^{-.5t} \cos (1.102t - .42547) \qquad \text{(IV-13)}$$

Equation (IV-13) is basically that of a second order system, which was expected since the example equation is basically second order, with a non-linear term. A satisfactory approximation should start at zero and follow a damped sinusoidal path to some steady-state value. The approximation here does have a damped sinusoidal characteristic and proceeds to a steady-state value that is the same as the value predicted. To meet the requirement that the solution begin at zero it is necessary that the second term in the approximate solution be equal to zero, or that

$$.72 \cos (-.42547) = .6825$$

It will be shown that this is very nearly the case.

2. Solution To The General Equation.

The characteristic equation for the general case, equation (IV-11), was found to be

$$X(s) = \frac{1}{s(s^2 + s + 1 + Ax_f^2)}$$

For the same values of A as in Part A the time solutions were found to be as follows:

A	x(t)
.5	$.7708 - .8468e^{-.5t}\cos (1.024t - .45378)$
.9	$.6956 - .7642e^{-.5t}\cos (1.091t - .42980)$
1.0	$.6825 - .7200e^{-.5t}\cos (1.102t - .42547)$
1.1	$.6689 - .7526e^{-.5t}\cos (1.116t - .41384)$
1.5	$.6300 - .6780e^{-.5t}\cos (1.155t - .40833)$
2.0	$.5896 - .6400e^{-.5t}\cos (1.201t - .42927)$

For all values of A the approximate solution oscillated slightly slower than the computer solution, but both solutions had the same number of oscillations. The difference between peak overshoots was in all cases much less than 10%, as was the difference between final values. As can be seen in Tables IV-8 through IV-13, the initial values of the approximate solutions were indeed very close to zero, the worst case being when A = 1.0 and A = 1.1, when the value of the approximate solution at t = 0 was -.00217.

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00097	.00097
0.5	.10440	.11243	.00803
1.0	.33989	.33845	.00144
1.5	.60451	.58305	.02146
2.0	.81756	.77810	.03946
2.5	.92862	.89458	.03404
2.8	.94273*	.92666	.02607
3.1	.92556	.93484*	.00928
3.5	.87331	.91794	.04463
4.0	.79738	.87134	.07396
4.5	.74154	.81807	.07653
5.0	.71795	.77401	.05706
5.2	.71687#	.76086	.04399
6.0	.74152	.73543#	.00509
6.5	.76366	.73807	.02559
7.0	.77966	.74757	.03209
7.6	.78646*	.76130	.02516
8.0	.78467	.76901	.01566
8.5	.77878	.77546	.00332
9.2	.77029	.77844*	.00815
9.9	.76646	.77660	.01014

TABLE IV-8
A = 0.5

* maximum
# minimum

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00090	.00090
0.5	.10440	.10442	.00002
1.0	.33956	.32977	.00979
1.5	.59980	.56705	.03275
2.0	.79372	.74632	.04740
2.5	.*86480**	.84110	.02370
2.9	.83921	.*86018**	.02097
3.5	.73951	.82436	.08485
4.0	.66772	.76820	.09048
4.6	.*63470*#	.70505	.07035
5.0	.64183	.67630	.03447
5.8	.68537	.*65663*#	.02874
6.5	.71275	.66807	.04468
6.9	.*71623**	.67896	.03727
7.5	.70913	.69378	.00535
8.0	.69914	.70162	.00221
8.6	.69123	.*70484**	.01361
9.0	.*68950*#	.70408	.01458
9.9	.69327	.69876	.00549

TABLE IV-9
A = 0.9

* maximum
# minimum

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	-.00217	.00217
0.5	.10439	.10442	.00003
1.0	.33947	.32977	.00970
1.5	.59862	.56705	.03157
2.0	.78797	.74632	.04165
2.5	.85002*	.84110	.00912
2.9	.81757	.86018*	.04261
3.5	.71375	.82436	.11061
4.0	.64515	.76820	.12305
4.5	.61960#	.71417	.09457
5.0	.63091	.67630	.04539
5.8	.67750	.65663#	.02087
6.7	.70314#	.67342	.02972
7.5	.69274	.69378	.00104
8.0	.68267	.70162	.01895
8.6	.67578	.70484*	.02906
9.0	.67525	.70408	.02883
9.9	.68057	.69876	.01819

TABLE IV-10
A = 1.0

* maximum
# minimum

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	-.00271	.00271
0.5	.10440	.08885	.01555
1.0	.33939	.32040	.01899
1.5	.59747	.56024	.03723
2.0	.78231	.73669	.04562
2.5	.83615*	.82438	.01177
2.5	.83615	.82438	.01177
2.8	.81156	.83754*	.02598
3.5	.69032	.79173	.10141
4.0	.62529	.73153	.11324
4.5	.60557#	.67115	.07158
5.0	.62180	.64182	.02002
5.6	.65910	.62765#	.03146
6.0	.67941	.63143	.04798
6.6	.69127*	.64712	.04415
7.0	.68784	.65890	.02894
7.5	.67746	.67071	.00674
8.0	.66750	.67735	.00015
8.6	.66214	.67882*	.01668
8.7	.66202#	.67853	.01831
9.5	.66608	.67360	.00725
9.9	.66911	.67078#	.00167

TABLE IV-11
A = 1.1

* maximum
# minimum

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00019	.00019
0.5	.10440	.10951	.00511
1.0	.33906	.32818	.01088
1.5	.59286	.55181	.04105
2.0	.76043	.71103	.04940
2.3	.79151*	.76455	.02696
2.7	.76030	.78966*	.02936
3.5	.61477	.73391	.11904
4.2	.56072#	.65212	.00860
5.0	.59678	.59613	.00065
5.5	.63090	.58906#	.04184
6.2	.65221*	.60276	.04945
7.0	.63870	.62639	.01231
7.5	.62622	.63621	.00999
8.2	.61907#	.64051*	.02144
9.0	.62442	.63637	.01195
9.5	.62902	.63245	.00343
9.9	.63112	.62985#	.00127

TABLE IV-12
A = 1.5

* maximum
# minimum

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00011	.00011
0.5	.10439	.09845	.00594
1.0	.33865	.31138	.02727
1.5	.58720	.52996	.05724
2.2	.74929*	.71719	.03210
2.6	.71288	.74679*	.03391
3.0	.63342	.71233	.07891
3.5	.54876	.67929	.13053
4.0	.51945#	.61830	.09885
4.5	.53894	.57207	.03313
5.0	.57814	.54967	.07047
5.3	.59863	.54708#	.05155
5.9	.59863*	.55481	.05769
6.5	.60512	.57821	.02691
7.8	.57921#	.60105	.02184
7.9	.57947	.60111*	.02611
8.5	.58527	.59816	.01289
9.0	.59100	.59371	.00271
9.7	.59385*	.58848#	.00537

TABLE IV-13
A = 2.0

* maximum
\# minimum

V. CONCLUSION

The specific object of the methods developed was to find a mathematically simple procedure with which to approximate the solution to certain non-linear differential equations. The only requirement was that the steady-state (final) value of the system variable could be obtained. This requirement implies a non-oscillatory steady-state, where all derivatives are zero. The best results were obtained in the case where there was an applied forcing function (non-homogeneous equation). The nature of the solution does not limit this function to a unit step, although either a constant input or one which decays to zero in the steady-state are the only inputs which will permit a final value of the system variable to be found. For the discharging capacitor example, with no input applied, the transient response was not very accurate, although the general path followed and the final value obtained were essentially the same as for the exact solution.

The usefulness of the final value method of approximation lies in its ready use by the engineer. The mathematics required is minimal and the solutions obtainable are sufficiently close to the exact solution to allow its use in analyzing non-linear systems which have a constant steady-state.

TIME (SEC)	COMPUTER SOLUTION	APPROXIMATE SOLUTION	ERROR
0.0	.00000	.00011	.00011
0.5	.10439	.09845	.00594
1.0	.33865	.31138	.02727
1.5	.58720	.52996	.05724
2.2	.74929*	.71719	.03210
2.6	.71288	.74679*	.03391
3.0	.63342	.71233	.07891
3.5	.54876	.67929	.13053
4.0	.51945#	.61830	.09885
4.5	.53894	.57207	.03313
5.0	.57814	.54967	.07047
5.3	.59863	.54708#	.05155
5.9	.59863*	.55481	.05769
6.5	.60512	.57821	.02691
7.8	.57921#	.60105	.02184
7.9	.57947	.60111*	.02611
8.5	.58527	.59816	.01289
9.0	.59100	.59371	.00271
9.7	.59385*	.58848#	.00537

TABLE IV-13
A = 2.0

* maximum
# minimum

V. CONCLUSION

The specific object of the methods developed was to find a mathematically simple procedure with which to approximate the solution to certain non-linear differential equations. The only requirement was that the steady-state (final) value of the system variable could be obtained. This requirement implies a non-oscillatory steady-state, where all derivatives are zero. The best results were obtained in the case where there was an applied forcing function (non-homogeneous equation). The nature of the solution does not limit this function to a unit step, although either a constant input or one which decays to zero in the steady-state are the only inputs which will permit a final value of the system variable to be found. For the discharging capacitor example, with no input applied, the transient response was not very accurate, although the general path followed and the final value obtained were essentially the same as for the exact solution.

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13. ABSTRACT

The analysis of non-linear systems has frequently been a major problem to the engineer. The solution of system equations often requires either a computer or relatively complex numerical techniques. A straight-forward, relatively simple method is proposed herein which permits the engineer to satisfactorily approximate the exact solution to certain non-linear differential equations.

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Approximate solution						
Non-linear differential equation						
Laplace transformation						